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ON THE CONVERGENCE OF BINET'S FACTORIAL SERIES FOR LOG $\Gamma(s)$ AND $\psi(s)$.*

By T. H. Gronwall.

It is the purpose of the present note to extend to the boundary of the region of convergence the discussion of the series in question given in § 14 of Jensen's "An elementary exposition of the theory of the gamma function."†

Writing (J 13, 14 and J 27, 28)

$$\begin{aligned}\omega(s) &= \log\Gamma(s) - (s - \tfrac{1}{2})\log s + s - \log\sqrt{2\pi} \\ &= \sum_0^\infty\left[(s + \nu + \tfrac{1}{2})\log\frac{s + \nu + 1}{s + \nu} - 1\right], \\ \omega^*(s) &= \log s - \psi(s) \\ &= \sum_0^\infty\left[\frac{1}{s + \nu} - \log\left(1 + \frac{1}{s + \nu}\right)\right],\end{aligned}$$

the four Binet series are (J 53, 54, 55, 56)

$$\omega(s) = \sum_1^\infty \frac{k_n}{n(s)_n}, \qquad k_n = \int_{-1}^0 (t + \tfrac{1}{2})(t)_n dt; \tag{1}$$

$$\omega(s) = \sum_1^\infty \frac{k_n}{n(s + 1)_n}, \qquad k_n = \int_0^1 (t - \tfrac{1}{2})(t)_n dt; \tag{2}$$

$$\omega^*(s) = \sum_1^\infty \frac{k_n}{n(s)_n}, \qquad k_n = -\int_{-1}^0 (t)_n dt; \tag{3}$$

$$\omega^*(s) = \sum_1^\infty \frac{k_n}{n(s + 1)_n}, \qquad k_n = \int_0^1 (t)_n dt; \tag{4}$$

where

$$(s)_n = s(s + 1)\cdots(s + n - 1) = \frac{\Gamma(s + n)}{\Gamma(s)}. \tag{5}$$

It is shown in J § 14 that all four Binet series are *absolutely* and *uniformly* convergent for $\Re(s) \geqq \epsilon > 0$, but divergent for $\Re(s) < 0$, where ϵ is arbitrarily small.

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* Read before the American Mathematical Society, Sept. 2, 1916.

† Authorized translation, with additional notes, by T. H. Gronwall, these *Annals*, ser. 2, vol. 17 (1916), pp. 124–166. References to formulas in this paper will be given thus: (J53).

I now propose to show that

The series (1) *to* (4) *are all* UNIFORMLY, *and* (1) *and* (3) ABSOLUTELY *convergent for* $|s| \geqq \epsilon$, $\Re(s) \geqq 0$, *where* ϵ *is arbitrarily small, but* (2) *and* (4) *are not* ABSOLUTELY *convergent for* $\Re(s) = 0$.

The proof will consist in obtaining comparison series by means of asymptotic expressions for the various coefficients k_n.

From the first part of (5) and Euler's product for $\Gamma(s)$ (J 3) it is seen at once that

$$(t)_n = \frac{(n-1)!\,n^t}{\Gamma(t)} \cdot \prod_{\nu=n}^{\infty} \frac{\left(1+\frac{1}{\nu}\right)^t}{1+\frac{t}{\nu}},$$

and from the expansion

$$t \log\left(1+\frac{1}{\nu}\right) - \log\left(1+\frac{t}{\nu}\right) = \frac{t}{\nu} - \frac{t}{2\nu^2} + \frac{t}{3\nu^3} - \cdots + (-1)^{\mu+1}\frac{t}{\mu\nu^\mu}$$
$$+ \cdots - \left[\frac{t}{\nu} - \frac{t^2}{2\nu^2} + \frac{t^3}{3\nu^3} - \cdots + (-1)^{\mu+1}\frac{t^\mu}{\mu\nu^\mu} + \cdots\right]$$

it follows that for $-1 \leqq t \leqq 1$

$$\left| t\log\left(1+\frac{1}{\nu}\right) - \log\left(1+\frac{t}{\nu}\right)\right| \leqq \frac{1}{\nu^2} + \frac{1}{\nu^3} + \frac{1}{\nu^4} + \cdots = \frac{1}{\nu(\nu-1)},$$

$$\left|\sum_{\nu=n}^{\infty}\left[t\log\left(1+\frac{1}{\nu}\right) - \log\left(1+\frac{t}{\nu}\right)\right]\right| \leqq \sum_{\nu=n}^{\infty}\frac{1}{\nu(\nu-1)}$$
$$= \frac{1}{n-1} < \frac{2}{n} \quad \text{for } n > 2,$$

$$\left|\prod_{\nu=n}^{\infty}\frac{\left(1+\frac{1}{\nu}\right)^t}{1+\frac{t}{\nu}} - 1\right| < |e^{2/n} - 1| < \frac{c_1}{n},$$

where c_1 is some constant independent of n, so that finally

$$(6) \qquad (t)_n = \frac{(n-1)!\,n^t}{\Gamma(t)}\left(1 + O\left(\frac{1}{n}\right)\right).^*$$

Now, by (6), (1) and (3) have a k_n of the form

$$k_n = (n-1)!\int_{-1}^{0} F(t)[n^t + O(n^{t-1})]dt,$$

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[*] The notation $f(x) = O(g(x))$, now quite commonly used in asymptotic calculations, signifies that there exist an $A > 0$ and an x_0 sufficiently large such that $|f(x)| < Ag(x)$ for $x > x_0$. It is evident from this definition that if $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$, then $f_1(x)f_2(x) = O(g_1(x)g_2(x))$, and $f_1(x) + f_2(x) = O(g_1(x))$ if $g_1(x) \geqq g_2(x)$ for $x > x_0$.

where $F(t) = (t + \frac{1}{2})/\Gamma(t)$ or $-1/\Gamma(t)$ is holomorphic in the interval of integration and vanishes at both its ends; hence, integrating twice by parts

$$\int_{-1}^{0} F(t)n^t dt = \left[-\frac{F'(t)n^t}{(\log n)^2} \right]_{-1}^{0} + \frac{1}{(\log n)^2} \int_{-1}^{0} F''(t)n^t dt,$$

and by the first theorem of the mean

$$\left| \int_{-1}^{0} F''(t)n^t dt \right| < \int_{-1}^{0} | F''(t) | dt,$$

so that

$$\int_{-1}^{0} F(t)n^t dt = O\left(\frac{1}{(\log n)^2} \right).$$

By a further application of the first theorem of the mean

$$\int_{-1}^{0} F(t)O(n^{t-1})dt = O\left(\frac{1}{n} \right),$$

so that finally

$$k_n = O\left(\frac{(n-1)!}{(\log n)^2} \right).$$

Consequently, the series (1) and (3) are absolutely and uniformly convergent whenever this is the case with the series

$$\sum_{2}^{\infty} \left| \frac{(n-1)!}{n(\log n)^2 (s)_n} \right|.$$

But for $s = x + iy$, $x \geqq 0$ and $\nu > 0$, we have $|s + \nu|^2 = (\nu + x)^2 + y^2 \geqq \nu^2$, whence by (5)

$$|(s)_n| \geqq |s| \cdot 1 \cdot 2 \cdots (n-1), \tag{7}$$

so that, for $x = \Re(s) \geqq 0$ and $|s| \geqq \epsilon$,

$$\sum_{2}^{\infty} \left| \frac{(n-1)!}{n(\log n)^2 (s)_n} \right| \leqq \frac{1}{\epsilon} \sum_{2}^{\infty} \frac{1}{n(\log n)^2},$$

and the latter series being convergent, the first half of our proposition follows.

In the series (2) and (4), we have, by (6), a k_n of the form

$$k_n = (n-1)! \int_{0}^{1} F(t)[n^t + O(n^{t-1})]dt,$$

where $F(t)$ is holomorphic in the integration interval and $F(0) = 0$, but $F(1) \neq 0$. Treating this integral in the same manner as before, we find

$$k_n = \frac{n!F(1)}{\log n} + O\left(\frac{n!}{(\log n)^2} \right).$$

The same argument as before shows that the series arising from the second term of k_n in (2) and (4) is absolutely and uniformly convergent for $\Re(s) \geqq 0$, so that now our comparison series becomes

$$\sum_2^{\infty} \frac{(n-1)!}{\log n \cdot (s+1)_n}. \tag{8}$$

To this series, we shall apply Dirichlet's convergence test.* We begin by showing that

$$A_n(s) = \sum_{\nu=1}^{n} \frac{(\nu-1)!}{(s+1)_\nu}$$

is uniformly bounded for $|s| \geqq \epsilon$, $\Re(s) \geqq 0$. From (5) we obtain the identity

$$\frac{(\nu-1)!}{(s+1)_{\nu-1}} - \frac{\nu!}{(s+1)_\nu} = \frac{(\nu-1)!}{(s+1)_{\nu-1}}\left(1 - \frac{\nu}{s+\nu}\right) = s \cdot \frac{(\nu-1)!}{(s+1)_\nu},$$

and making $\nu = 1, 2, \cdots, n$ and adding

$$A_n(s) = \frac{1}{s}\left(1 - \frac{n!}{(s+1)_n}\right).$$

For $\Re(s) \geqq 0$, we have, in analogy to (7), $|(s+1)_n| \geqq n!$, and consequently

$$|A_n(s)| \leqq \frac{1}{|s|}(1+1) \leqq \frac{2}{\epsilon} \quad \text{for} \quad |s| \geqq \epsilon \quad \text{and} \quad \Re(s) \geqq 0.$$

From the identity

$$\sum_{\nu=n}^{n+p} \frac{(\nu-1)!}{\log \nu (s+1)_\nu} = \sum_{\nu=n}^{n+p} \frac{A_\nu(s) - A_{\nu-1}(s)}{\log \nu}$$
$$= \sum_{\nu=n}^{n+p-1} A_\nu(s)\left(\frac{1}{\log \nu} - \frac{1}{\log(\nu+1)}\right) + \frac{A_{n+p}(s)}{\log(n+p)} - \frac{A_{n-1}(s)}{\log n}$$

we now obtain, since $1/\log \nu$ decreases as ν increases,

$$\left|\sum_{\nu=n}^{n+p} \frac{(\nu-1)!}{\log \nu (s+1)_\nu}\right| \leqq \frac{2}{\epsilon} \sum_{\nu=n}^{n+p-1}\left(\frac{1}{\log \nu} - \frac{1}{\log(\nu+1)}\right) + \frac{2}{\epsilon} \cdot \frac{1}{\log(n+p)} + \frac{2}{\epsilon} \cdot \frac{1}{\log n}$$
$$= \frac{4}{\epsilon} \cdot \frac{1}{\log n},$$

and the last expression is independent of s and may be made as small as we please by taking n sufficiently large. Thus (2) and (4) are uniformly convergent when $|s| \geqq \epsilon$, $\Re(s) \geqq 0$. Regarding the absolute convergence of (8) we observe that, since $\Gamma(s) = \lim_{n \to \infty} (n-1)! n^s/(s)_n$, we have for a fixed s and an n sufficiently large

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* Journ. de Math., ser. 2, vol. 7, pp. 253–255.

$$\frac{1}{2}\left|\frac{\Gamma(s+1)}{n^{s+1}}\right| < \left|\frac{(n-1)!}{(s+1)_n}\right| < 2 \cdot \left|\frac{\Gamma(s+1)}{n^{s+1}}\right|,$$

so that the series of absolute values in (8) will converge or diverge with

$$\sum_2^\infty \frac{1}{\log n \, |n^{s+1}|} = \sum_2^\infty \frac{1}{n^{1+\Re(s)} \cdot \log n}.$$

Hence (8) is absolutely convergent when $\Re(s) > 0$, but not when $\Re(s) = 0$.